Vector Integration

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Line Integral: The integration ~~of F~~ along a curve c joining the points A & B in the force field f represents work done. This is physical interpretation of line integral

$$\boxed{\text{A line Integral} = \int_c \bar{F}\cdot d\bar{r}}$$

let $\bar{F} = F_1 i + F_2 j + F_3 k$ & $d\bar{r} = i\,dx + j\,dy + k\,dz$ then

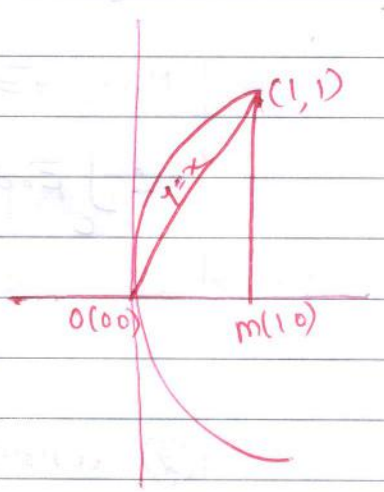
$$\bar{F}\cdot d\bar{r} = (F_1 i + F_2 j + F_3 k)\cdot(i\,dx + j\,dy + k\,dz)$$
$$= F_1 dx + F_2 dy + F_3 dz$$

$$\text{A line integral} = \int_c \bar{F}\cdot d\bar{r} = \int_c F_1 dx + F_2 dy + F_3 dz$$

Example 1: We shall now obtain $\int_c \bar{F}\cdot d\bar{r}$ for $\bar{F} = x^2 i + xy j$ in two cases (1) c is the curve $y^2 = x$ joining $(0,0)$ & $(1,1)$ is
(2) c is the curve $y = x$ joining the same points.

$$\int_c \bar{F}\cdot d\bar{r} = \int_c (x^2 i + xy j)\cdot(i\,dx + j\,dy)$$

$$= \int_c x^2 dx + xy\,dy$$



(1) consider the parabolic path op joining $(0,0)$ & $(1\ 1)$

Eqn of parabola is $y^2 = x$

$$2y\,dy = dx$$

$$\therefore \int_c x^2 dx + xy\,dy = \int_0^1 y^4 \cdot 2y\,dy + y^2 \cdot y\,dy$$

$$= \int_0^1 (2y^5 + y^3)\,dy = \left[\frac{2y^6}{6} + \frac{y^4}{4}\right]_0^1$$

$$= \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$$

ii) consider the straight line $y=x$ joining $(0,0)$ & $(1,1)$

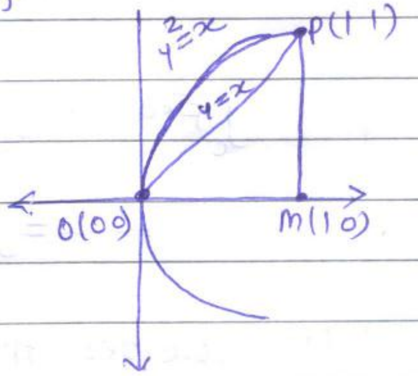
$$\therefore \int_c x^2dx + xy\,dy = \int_0^1 x^2dx + x x\,dx = \int_0^1 2x^2dx = 2\frac{x^3}{3}\Big]_0^1 = \frac{2}{3}$$

Thus we get different values for different paths.

The value of line integral $\int_{CA}^{B} \bar{F}\cdot d\bar{r}$ depends upon the curve C joining the points A & B. ie we get different values of line integrals along the different curves joining the same points. If vector field $\bar{F}$ is irrotational ie conservative then the value of line integral $\int_{CA}^{B}\bar{F}\cdot d\bar{r}$ is independent of the path joining the same points. To illustrate consider the following example.

let $\bar{F} = x^2 i + y^2 j$ which is irrotational. Let us find $\int_c \bar{F}\cdot d\bar{r}$ along the three different paths joining the same points.

① consider the path $y^2=x$ joining $(0,0)$ & $(1,1)$.



$$\int_c \bar{F}\cdot d\bar{r} = \int_c x^2dx + y^2dy$$

Now $y^2=x \Rightarrow 2y\,dy = dx$

$$\therefore \int_c \bar{F}\cdot d\bar{r} = \int_0^1 y^4\, 2y\,dy + y^2dy$$

$$= \int_0^1 2y^5dy + y^2dy = \left[2\frac{y^6}{6} + \frac{y^3}{3}\right]_0^1 = \frac{1}{3}+\frac{1}{3} = \frac{2}{3}$$

② consider the path $y=x$ (straight line)

$$\int_c \bar{F}\cdot d\bar{r} = \int_c x^2dx + y^2dy \qquad [x=y, \ \therefore dx=dy]$$

$$= \int_0^1 x^2dx + x^2dx = \int_0^1 2x^2dx = 2\left[\frac{x^3}{3}\right]_0^1 = \frac{2}{3}$$

(iii) consider the path om and then mp

$$\therefore \int_C \bar{F}\cdot d\bar{r} = \int_{om} \bar{F}\cdot d\bar{r} + \int_{mp} \bar{F}\cdot d\bar{r}$$

$$= \int_{om} x^2dx + y^2dy + \int_{mp} x^2dx + y^2dy.$$

along om, $y=0$, $dy=0$
along mp, $x=1$ $dx=0$.

$$\int_C \bar{F}\cdot d\bar{r} = \int_0^1 x^2dx + \int_0^1 y^2dy = \left[\frac{x^3}{3}\right]_0^1 + \left[\frac{y^3}{3}\right]_0^1 = \frac{1}{3}+\frac{1}{3} = \frac{2}{3}$$

Hence the values $\int_C \bar{F}\cdot d\bar{r}$ come out to be the same for all the three paths joining $(0,0)$ and $(1,1)$ because $\bar{F}$ is conservative.

Ex 2. Evaluate $\int_C \bar{F}\cdot d\bar{r}$ for $\bar{F} = 3x^2 i + (2xz - y) j + z k$ along the following paths i) The straight line joining $(0,0,0)$ & $(2\,1\,3)$
ii) The curve $x = 2t^2$, $y = t$ $z = 4t^2 - t$ from $t=0$ to $t=1$
iii) Along the curve defined by $x^2 = 4y$, $3x^3 = 8z$ from $x = 0$ to $x = 2$.

Sol i) $\int_C \bar{F}\cdot d\bar{r} = \int_C 3x^2dx + (2xz - y)dy + z\,dz$

along the straight line joining $(0,0,0)$ and $(2,1,3)$ which is given by $\frac{x-0}{2-0} = \frac{y-0}{1-0} = \frac{z-0}{3-0}$ ie $\frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t$

ie $x = 2t$, $y = t$, $z = 3t$ and $dx = 2dt$, $dy = dt$, $dz = 3dt$ and t varies from 0 to 1 along the path.

$$\int_C \bar{F}\cdot d\bar{r} = \int_0^1 3(4t^2)\,2dt + (12t^2 - t)dt + 3t\cdot 3dt$$

$$= \int_0^1 (24t^2 + 12t^2 - t + 9t)\,dt = \int_0^1 (36t^2 + 8t)\,dt$$

$$= \left[36\frac{t^3}{3} + 8\frac{t^2}{2}\right]_0^1 = \frac{36}{3} + \frac{8}{2} = 16$$

ii) along the curve $x = 2t^2$, $y = t$, $z = 4t^2 - t$

$dx = 4t\,dt$, $dy = dt$, $dz = (8t-1)\,dt$

$$\int_c \bar{F}\cdot d\bar{r} = \int_0^1 3(4t^4)4t\,dt + (16t^4 - 4t^3 - t)\,dt + (4t^2 - t)(8t-1)\,dt$$

$$= \int_0^1 (48t^5 + 16t^4 - 4t^3 - t + 32t^3 - 12t^2 + t)\,dt$$

$$= \int_0^1 (48t^5 + 16t^4 + 28t^3 - 12t^2]\,dt$$

$$= \left[48\frac{t^6}{6} + 16\frac{t^5}{5} + 28\frac{t^4}{4} - 12\frac{t^3}{3}\right]_0^1$$

$$= 8 + \frac{16}{5} + 7 - 4 = \frac{71}{5}$$

iii) Along the curve $x^2 = 4y$, $3x^3 = 8z$ from $x = 0$ to 2

The parametric equation may be taken as

$x = 2t$, $y = t^2$ $z = 3t^3$ from $t = 0$ to 1

$dx = 2dt$ $dy = 2t\,dt$ $dz = 9t^2\,dt$

$$\int_c \bar{F}\cdot d\bar{r} = \int_0^1 3(4t^2)2\,dt + (12t^4 - t^2)\,2t\,dt + 3t^3\cdot 9t^2\,dt$$

$$= \int_0^1 (24t^2 + 24t^5 - 2t^3 + 27t^5)\,dt$$

$$= \int_0^1 (51t^5 - 2t^3 + 24t^2)\,dt$$

$$= \left[51\frac{t^6}{6} - 2\frac{t^4}{4} + 24\frac{t^3}{3}\right]_0^1$$

$$= \frac{51}{6} - \frac{1}{2} + 8 = 16$$

Ex 3 Find the work done in moving a particle once round the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$, $z=0$ under the field of force given by

$$\bar{F} = (2x-y+z)\hat{i} + (x+y-z^2)\hat{j} + (3x-2y+4z)\hat{k}$$

is the field conservative ?

Work done $W = \int_c \bar{F}\cdot d\bar{r} = \int_c (2x-y+z)dx + (x+y-z^2)dy + (3x-2y+4z)dz$

where c is the arc of the ellipse.

Since $z=0$, $dz=0$

$$\therefore W = \int_c (2x-y)dx + (x+y)dy$$

Taking the parametric equation of ellipse $x=5\cos\theta$, $y=4\sin\theta$, $dx=-5\sin\theta\, d\theta$, $dy=4\cos\theta\, d\theta$

$$W = \int_0^{2\pi} (10\cos\theta - 4\sin\theta)(-5\sin\theta)d\theta + (5\cos\theta+4\sin\theta)(4\cos\theta)d\theta$$

$$= \int_0^{2\pi} [-50\sin\theta\cos\theta + 20\sin^2\theta + 20\cos^2\theta + 16\sin\theta\cos\theta]\, d\theta$$

$$= \int_0^{2\pi} 20(\sin^2\theta+\cos^2\theta)\, d\theta - 34\int_0^{2\pi} \sin\theta\cdot\cos\theta\cdot d\theta$$

$$= 20[\theta]_0^{2\pi} = 40\pi \qquad \because \int_0^{2\pi} \sin\theta\cdot\cos\theta\, d\theta = 0$$

Since the work done is not zero. the vector field is not conservative.

Example 4 Evaluate $\int \bar{F}\cdot d\bar{r}$ for $\bar{F} = (2y+3)\hat{i} + (xy)\hat{j} + (yz-x)\hat{k}$ along the following paths

(1) $x^2 = 2t^2$, $y = t$, $z = t^3$ from $t=0$ to 1

(2) the straight lines from $(0,0,0)$ to $(0,0,1)$ then to $(0,1,1)$ and then to $(2,1,1)$

(c) The straight line joining $(0\ 0\ 0)$ & $(3,1,1)$

Ans (I) $\dfrac{483\sqrt{2}+60}{140}$, (II) 10, (III) $\dfrac{71}{6}$

Ex 5

Evaluate $\int_c \bar{F}\cdot d\bar{r}$ for $\bar{F} = (2x+y)\hat{i} + (3y-x)\hat{j}$ and c is the curve

1) straight line joining $(0\ 0)$ & $(3\ 2)$

11) Along the path joining $(0\ 0)$ and $(2,0)$ and the ~~co~~, from $(2,0)$ to $(0,3)$

Soln:

$$\int_c \bar{F}\cdot d\bar{r} = \int_c (2x+y)\,dx + (3y-x)\,dy$$

(1) c is straight line joining $(0\ 0)$ & $(3,2)$

ie $\dfrac{x-0}{3} = \dfrac{y-0}{2}$ ie $\dfrac{x}{3} = \dfrac{y}{2} = t$

$x = 3t$, $y = 2t$

$dx = 3dt$ $\quad dy = 2dt$ from $t=0$ to $t=1$

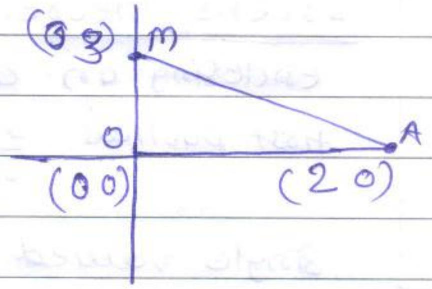
$$\int_c \bar{F}\cdot d\bar{r} = \int_0^1 (2(3t)+2t)\,3dt + (3(2t)-3t)\,2\,dt$$

$$= \int_0^1 (6t+2t)\,3dt + (6t-3t)\,2dt$$

$$= \int_0^1 (24t\,dt + 6t\,dt) = \left[30\frac{t^2}{2}\right]_0^1$$

$$= 15$$

(11) c is the path OAM.



Eqⁿ of AM is

$$\frac{x-2}{0-2} = \frac{y-0}{3-0}$$

ie $\frac{x-2}{-2} = \frac{y}{3} = t$

$x = 2-2t,\ y = 3t, \quad dx = -2dt \quad dy = 3dt$

From $t = 0$, to $t = +1$,

$$\therefore \int_C \bar{F}\cdot d\bar{r} = \int_{OA} \bar{F}\cdot d\bar{r} + \int_{AM} \bar{F}\cdot d\bar{r}$$

$$= \int_{OA} (2x+y)\,dx + (3y-x)\,dy + \int_{AM} (2x+y)\,dx + (3y-x)\,dy$$

along OA x varies from 0 to 2 & $y = 0$ $dy = 0$

$$\therefore I_1 = \int_{OA} (2x+y)\,dx + (3y-x)\,dy = \int_0^2 2x\,dx = \left[2\frac{x^2}{2}\right]_0^2 = 4$$

$$I_2 = \int_{AM} (2x+y)\,dx + (3y-x)\,dy = \int_0^1 [2(2-2t)+3t](-2dt)$$

$$+ [3(3t)-2+2t]\,3dt$$

$$= \int_0^1 [4-4t+3t](-2dt) + [9t-2+2t]\,3dt$$

$$= \int_0^1 [-8+8t-6t]\,dt + [27t-6+6t]\,dt$$

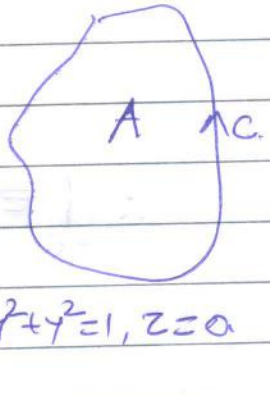
$$= \int_0^1 [35t-14]\,dt = \left[\frac{35t^2}{2} - 14t\right]_0^1$$

$$= \frac{35}{2} - 14 = \frac{35-28}{2} = \frac{7}{2}$$

$$\therefore \int_C \bar{F}\cdot d\bar{r} = 4 + \frac{7}{2} = \frac{15}{2}$$

Green's Theorem : Consider the closed curve c enclosing an area A. Let $u(x,y)$, $v(x,y)$ and their first partials $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ be continueous and single valued over the region bounded by the curve c then

$$\oint_C u\,dx + v\,dy = \iint_A \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$$



Example 1 Evaluate $\oint_C (\cos y\,\hat{i} + x(1-\sin y)) \cdot d\bar{r}$ for the closed curve which is given by $x^2+y^2=1, z=0$

Soln

$$I = \oint_C \cos y\,dx + x(1-\sin y)\,dy$$

Here $u = \cos y$, $v = x(1-\sin y)$

$$\frac{\partial u}{\partial y} = -\sin y, \quad \frac{\partial v}{\partial x} = 1-\sin y$$

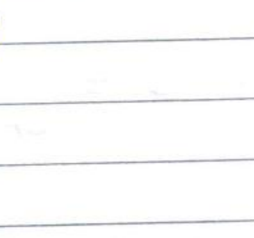
$$\therefore I = \oint_C \cos y\,dx + x(1-\sin y)\,dy$$

$$= \oint_C u\,dx + v\,dy = \iint_A \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$$

$$= \iint_A (+\sin y + 1 + \sin y)\,dx\,dy$$

where A is the area of circle $x^2+y^2=1$

$$= \iint_A dx\,dy = \pi$$



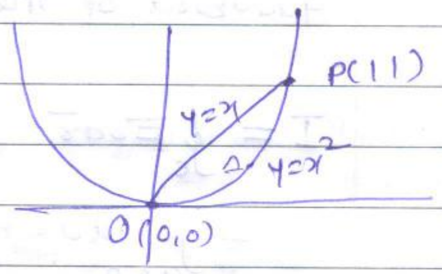
$$= \oint\int_{r=0\ \theta=0}^{1\ \ 2\pi}$$

or $I = \iint_A dx\,dy = \int_0^1\int_0^{2\pi} r\,dr\,d\theta = \pi$

Example 2 Verify Green's Theorem for the field $F = x^2 i + xy j$ over the region R enclosed by $y = x^2$ and then line $y = x$

By Greens theorem $\oint_C u\,dx + v\,dy = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$

$LHS = \oint_C x^2 dx + xy\,dy$



$= \int_{OP} (x^2 dx + xy\,dy) + \int_{PAO} x^2 dx + xy\,dy$

$= I_1 + I_2$

$I_1 = \int_{OP} x^2 dx + xy\,dy \quad , \quad y = x, \quad dy = dx$

$= \int_0^1 x^2 dx + x^2 dx = 2\left[\frac{x^3}{3}\right]_0^1 = \frac{2}{3}$

$I_2 = \int_{PAO} x^2 dx + xy\,dy \quad y = x^2, \quad dy = 2x\,dx$

$= \int_1^0 x^2 dx + x \cdot x^2\, 2x\,dx = \int_1^0 x^2 dx + 2x^4 dx$

$= \left[\frac{x^3}{3} + 2\frac{x^5}{5}\right]_1^0 = -\frac{1}{3} - \frac{2}{5} = -\frac{11}{15}$

$LHS = I_1 + I_2 = \frac{2}{3} - \frac{11}{15} = -\frac{1}{15}$

Now

$RHS = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$

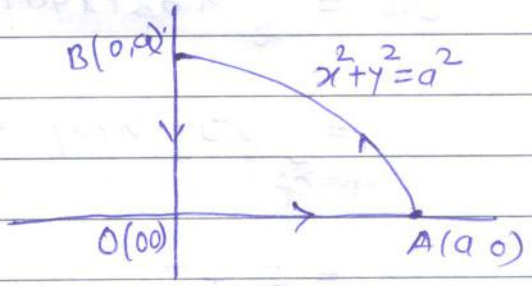
$v = xy, \quad u = x^2$

$RHS = \int_{x=0}^{x=1} \int_{y=x^2}^{y=x} (y - 0)\,dx\,dy = \int_0^1 \left[\frac{y^2}{2}\right]_{x^2}^{x} = \frac{1}{2}\int_0^1 (x^2 - x^4)\,dx$

$= \frac{1}{2}\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = \frac{1}{2}\left[\frac{1}{3} - \frac{1}{5}\right] = \frac{1}{15}$

Hence proved.

Example 3 Verify Greens theorem for $\bar{F} = x\hat{i} + y^2\hat{j}$ over the first quadrant of the circle $x^2+y^2=a^2$.



$I = \oint_C \bar{F} \cdot d\bar{r}$

$= \int_{OAB} x\,dx + y^2\,dy$

By Greens theorem

$I = \int_C P\,dx + Q\,dy = \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

$Q = y^2, \qquad P = x$

$\frac{\partial Q}{\partial x} = 0, \qquad \frac{\partial P}{\partial y} = 0$

$\therefore \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = 0$

$\int_C x\,dx + y^2\,dy$, where c is the path OABO

along OA $y=0$, $dy=0$, $I_1 = \int_0^a x\,dx = \frac{a^2}{2}$

along arc AB, $x = a\cos\theta$, $y = a\sin\theta$

$dx = -a\sin\theta\,d\theta, \quad dy = a\cos\theta\,d\theta$

$I_2 = \int_{AB} x\,dx + y^2\,dy = \int_0^{\pi/2} \left(-a^2\sin\theta\cos\theta + a^3\sin^2\theta\cdot\cos\theta\right) d\theta$

$= -a^2\left[-\frac{1\cdot 1}{2} + \frac{1\cdot 1}{3\cdot 1}\,a\right] = \frac{-a^2}{2} + \frac{a^3}{3}$

$I_3 = \int_{BO} y^2\,dy$ as $dx = 0$

$$= \left[\frac{y^3}{3}\right]_a^0 = -\frac{a^3}{3}$$

$$I = \frac{a^2}{2} - \frac{a^2}{2} + \frac{a^3}{3} - \frac{a^3}{3} = 0$$

Hence Greens Theorem is verified

Example 4 Using Green's theorem, show that the area bounded by a simple closed curve C is given by $\frac{1}{2}\int x\,dy - y\,dx$. Hence find the area of the ellipse $x = a\cos\theta,\ y = b\sin\theta$.

Sol By Green's Theorem $\oint_C p\,dx + \phi\,dy = \iint_A \left(\frac{\partial \phi}{\partial x} - \frac{\partial p}{\partial y}\right) dx\,dy$

Comparing ~~with~~ $\oint p\,dx + \phi\,dy$ with $\frac{1}{2}\int x\,dy - y\,dx$

We get $p = -\frac{y}{2} \quad \phi = \frac{x}{2}$

$$\frac{\partial p}{\partial y} = -\frac{1}{2}, \quad \frac{\partial \phi}{\partial x} = \frac{1}{2}$$

$$\therefore \frac{1}{2}\oint x\,dy - y\,dx = \iint_A \left(\frac{1}{2} + \frac{1}{2}\right) dx\,dy = \iint_A dx\,dy$$

ie area bounded by the closed curve

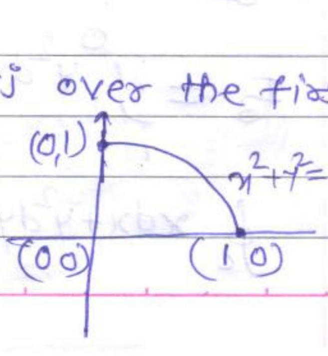
In case of ellipse $x = a\cos\theta,\ y = b\sin\theta$

$$\frac{1}{2}\oint_C x\,dy - y\,dx = \frac{1}{2}\int (a\cos\theta\, b\cos\theta + b\sin\theta \cdot a\sin\theta)\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} ab(\cos^2\theta + \sin^2\theta)\,d\theta = \frac{ab}{2}[\theta]_0^{2\pi}$$

$= \pi ab.$ Which is the area of the ellipse

Example 5 verify Green's theorem for $\bar{F} = xi + y^2 j$ over the first quadrant of the circle $x^2 + y^2 = 1$



Sol:

$$\oint_C u\,dx + v\,dy = \iint_A \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$$

Now

$$\oint_C \bar{F}\cdot d\bar{r} = \oint_C x\,dx + y^2\,dy, \quad u = x,\ v = y^2$$

$$= \int_{OA} x\,dx + y^2 dy + \int_{AB} x\,dx + y^2 dy + \int_{BO} x\,dx + y^2 dy$$

$$= I_1 + I_2 + I_3$$

along OA, $y = 0$, $dy = 0$, x varies from 0 to 1

$$I_1 = \int_0^1 x\,dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}$$

AB is arc of circle $x^2 + y^2 = 1$,

put $x = \cos\theta$, $y = \sin\theta$

$dx = -\sin\theta$ $\quad dy = \cos\theta\,d\theta$

$$I_2 = \int_0^{\pi/2} \cos\theta(-\sin\theta)\,d\theta + \sin^2\theta\cdot\cos\theta\,d\theta$$

$$= \int_0^{\pi/2} [-\sin\theta + \sin^2\theta]\cos\theta\,d\theta$$

put $\sin\theta = t$, $\quad \cos\theta\,d\theta = dt$

θ	0	$\pi/2$
t	0	1

$$= \int_0^1 [-t + t^2]\,dt = \left[-\frac{t^2}{2} + \frac{t^3}{3}\right]_0^1 = -\frac{1}{2} + \frac{1}{3} = -\frac{1}{6}$$

along BO, $x = 0$, $dx = 0$

$$I_3 = \int_1^0 y^2\,dy = \left[\frac{y^3}{3}\right]_1^0 = -\frac{1}{3}$$

$$\oint_C x\,dx + y^2\,dy = \frac{1}{2} - \frac{1}{6} - \frac{1}{3} = \frac{3-1-2}{6} = 0$$

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If $u = x,\ v = y^2$ $\quad \frac{\partial v}{\partial x} = 0, \quad \frac{\partial u}{\partial y} = 0$

$$\iint_A \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy = \iint_A (0-0)\, dx\,dy = 0$$

$$\therefore \oint_C u\,dx + v\,dy = \oint_C x\,dx + y^2 dy = \iint_A \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$$

Ex 6 If $\bar{F} = \frac{1}{x^2+y^2}[-y\hat{i} + x\hat{j}]$ then show that $\oint_C \bar{F} \cdot d\bar{r} = 2\pi$ where c is the circle containing origin.

Sol $I = \oint_C \bar{F} \cdot d\bar{r} = \oint_C \frac{-y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy = \oint_C \frac{-y\,dx + x\,dy}{x^2+y^2}$

changing variables to polar co-ordinates

$x = r\cos\theta, \quad y = r\sin\theta \quad \therefore x^2+y^2 = r^2$

$dx = -r\sin\theta\, d\theta \quad dy = r\cos\theta\, d\theta$ and θ varies from 0 to 2π.

$$I = \int_0^{2\pi} \frac{(-r\sin\theta)(-r\sin\theta\, d\theta) + (r\cos\theta)(r\cos\theta)d\theta}{r^2}$$

$$= \int_0^{2\pi} \frac{r^2\sin^2\theta + r^2\cos^2\theta}{r^2} d\theta = \int_0^{2\pi} d\theta = 2\pi.$$

Example 7: Find the work done in moving a particle once round the ellipse $\frac{x^2}{16} + \frac{y^2}{4} = 1,\ z = 0$. under the field of force given by $\bar{F} = (2x-y+z)\hat{i} + (x+y-z^2)\hat{j} + (3x-2y+4z)\hat{k}$.

Sol We know that

Workdone $= \oint_C \bar{F} \cdot d\bar{r} = \int_C (2x-y+z)\,dx + (x+y-z^2)dy + (3x-2y+4z)dz$

Here curve c: $\frac{x^2}{16} + \frac{y^2}{4} = 1, \quad z = 0$

parametric equation of given ellipse are given by

$x = 4\cos\theta \quad y = 3\sin\theta, \quad z = 0$

$dx = -4\sin\theta\, d\theta, \quad dy = 2\cos\theta \cdot d\theta \quad dz = 0.$

for complete circle θ varies from 0 to 2π

$$W = \int_0^{2\pi} [2(4\cos\theta) - (3\sin\theta) - 0](-4\sin\theta\, d\theta)$$

$$+ [4\cos\theta + 3\sin\theta](3\cos\theta \cdot d\theta)$$

$$= \int_0^{2\pi} (-9\sin\theta \cdot \cos\theta + 9\sin^2\theta + 9\cos^2\theta + 9\sin\theta \cdot \cos\theta)\, d\theta$$

$$= \int_0^{2\pi} [9\sin\theta \cdot \cos\theta + 9\sin^2\theta + 9\cos^2\theta]\, d\theta$$

$$= -9\int_0^{2\pi} \sin\theta \cdot \cos\theta\, d\theta + 9\int_0^{2\pi} d\theta = 18\pi.$$

Ex 8 Evaluate $\int_c \bar{F} \cdot d\bar{r}$ for $\bar{F} = 3x^2 i + (2xz - y) j + z k$ along the straight line joining the points $(0, 0, 0)$ and $(1, 2, 3)$

Let

$$I = \int_c \bar{F} \cdot d\bar{r} = \int_c 3x^2 dx + (2xz - y) dy + z\, dz \quad \text{—} ①$$

equation of line joining the points $(0\ 0\ 0)$ to $(1\ 2, 3)$ is straight line given by

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{3} = t \quad \Rightarrow x = t, \quad y = 2t, \quad z = 3t$$

$dx = dt, \quad dy = 2dt \quad dz = 3dt$ and t varies from 0 to 1. Using these terms in ① we get

$$I = \int_0^1 3t^2 dt + [2t(3t) - 2t]\, 2dt + (3t)\, 3dt$$

$$= \int_0^1 (15t^2 + 5t)\, dt = \left[15\frac{t^3}{3} + 5\frac{t^2}{2}\right]_0^1 = 5 + \frac{5}{2} = \frac{15}{2}$$

Ex 9 If $\overline{F} = (2x+y^2)i + (3y-4x)j$ then evaluate $\int_C \overline{F}\cdot d\overline{r}$ around the parabolic arc $y^2 = x$ joining $(0\ 0)$ & $(1,\ 1)$

$$I = \int_c \overline{F}\cdot d\overline{r} = \int_c (2x+y^2)\,dx + (3y-4x)\,dy \qquad — ①$$

Given curve $c: y^2 = x \quad \therefore y = \sqrt{x} \Rightarrow dy = \frac{1}{2\sqrt{x}}\,dx$

$$\therefore I = \int_0^1 (2x+x)\,dx + (3\sqrt{x}-4x)\,\frac{1}{2\sqrt{x}}\,dx = \frac{5}{3}$$

Ex 10 Evaluate $\int_c [12x^2y+y+z^2]\,i + (1+yz^3)\,j + (2z+3y^2z^2)\,k \cdot d\overline{r}$ along the curve $c: y^2+z^2=a^2,\ x=0$.

$$I = \int_c \overline{F}\cdot d\overline{r} = \int_c (2x^2y+y+z^2)\,dx + 2(1+yz^3)\,dy + (2z+3y^2z^2)\,dz$$

Given curve $c: y^2+z^2=a^2,\ x=0$ is a circle with centre at origin and radius is a in yz plane.

put $y = a\cos\theta \qquad z = a\sin\theta \qquad x = 0$

$dy = -a\sin\theta\,d\theta \qquad dz = a\cos\theta\,d\theta \qquad dx = 0$

θ varies from 0 to 2π

$$I = \int_0^{2\pi} [-2a\sin\theta - 2a^5\sin^4\theta\cdot\cos\theta + 2a^2\sin\theta\cdot\cos\theta + 3a^5\cos^5\theta]\,d\theta$$

$= 0$

$$\int_0^{2\pi} \sin^n\theta\,d\theta = 0 \quad \text{if } n = \text{odd}$$

$$\int_0^{2\pi} \sin^m\theta\,\cos^n\theta\,d\theta = 0 \quad \text{if } m \text{ or and } n \text{ are odd.}$$

Ex 11 Find the work done in moving a particle along $x = a\cos\theta$, $y = a\sin\theta$, $z = b\theta$ from $\theta = \pi/4$ to $\pi/2$ under the field $\overline{F} = (-3a\sin^2\theta\cdot\cos\theta)\,i + a(2\sin\theta - 3\sin^2\theta)\,j + b\sin 2\theta\,k$

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Workdone $W = \int_C \bar{F} \cdot d\bar{r}$

$$= \int_C -3a \sin^2\theta \cdot \cos\theta \, dx + a(2\sin\theta - 3\sin^2\theta) dy + b\sin 2\theta \, dz$$

Given $x = a\cos\theta \quad y = a\sin\theta \quad z = b\theta$

$dx = -a\sin\theta \, d\theta \quad dy = a\cos\theta \, d\theta \quad dz = b\,d\theta$

Using the values in ① we get

$$W = \int_{\pi/4}^{\pi/2} (-3a\sin^2\theta \cos\theta)(-a\sin\theta)d\theta + a(2\sin\theta - 3\sin^3\theta)(a\cos\theta \, d\theta) + b^2 \sin 2\theta \, d\theta$$

$$= \int_{\pi/4}^{\pi/2} [3a^2 \sin^3\theta \cdot \cos\theta + 2a^2 \sin\theta \cdot \cos\theta - 3a^2 \sin^3\theta \cdot \cos\theta + b^2 \sin 2\theta] \, d\theta$$

$$= \int_{\pi/4}^{\pi/2} (a^2 \sin 2\theta + b^2 \sin 2\theta) \, d\theta = \int_{\pi/4}^{\pi/2} (a^2 + b^2) \sin 2\theta \, d\theta$$

$$= -\frac{a^2+b^2}{2} \left[\cos\frac{2\pi}{2} - \cos\frac{2\pi}{4} \right]$$

$$= \frac{a^2+b^2}{2}$$

Ex 12 Find the work done by the force $\bar{F} = (x^2 - yz)i + (y^2 - 2x)j + (z^2 - xy)k$ in taking a particle from (1 1 1) to (3, -5 7)

Here path joining (1 1 1) to (3 -5 7) is not given. Therefore we check whether the given vector field is irrotational or not

$$\nabla\times\overline{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2-yz & y^2-xz & z^2-xy \end{vmatrix} = 0$$

$\therefore$ $\overline{F}$ is irrotational

There exist a scalar potential ϕ such that $\overline{F} = \nabla\phi$

$$\overline{F}\cdot d\overline{r} = \nabla\phi\cdot d\overline{r} = d\phi$$

$$\text{Workdone} = \int_{(1,1,1)}^{(3,-5,7)} \overline{F}\cdot d\overline{r} = \int_{(1,1,1)}^{(3,-5,7)} (x^2-yz)\,dx + (y^2-zx)\,dy + (z^2-yx)\,dz$$

Since $\overline{F}$ is irrotational, integral on RHS is an exact differential eqn

$$W = \left[\int_{y,z} (x^2-yz)\,dx + \int_{z=\text{const}} y^2\,dy + \int z^2\,dz\right]_{(1,1,1)}^{(3,-5,7)}$$

$$= \left[\frac{x^3}{3} - xyz + \frac{y^3}{3} + \frac{z^3}{3}\right]_{(1,1,1)}^{(3,-5,7)} = \frac{560}{3}$$

Ex 13 Find the work done in moving a particle from $(0,1,-1)$ to $(\pi/2, -1, 2)$ in a force field $\overline{F} = (y^2\cos x + z^3)\,i + (2y\sin x - 4)\,j + (3xz^2+2)\,k$

Here the path joining given two points is not mentioned therefore check whether the given vector field is irrotational or not.

consider

$$\nabla\times\overline{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2\cos x + z^3 & 2y\sin x - 4 & 3xz^2+2 \end{vmatrix} = 0$$

$\therefore \bar{F}$ is irrotational.

Workdone $W = \int_{(0\ 1\ -1)}^{(\pi/2\ -1\ 2)} \bar{F}\cdot d\bar{r}$

$$= \int_{(0\ 1\ -1)}^{(\pi/2\ -1\ 2)} (y^2\cos x + z^3)\,dx + (2y\sin x - 4)\,dy + (3xz^2+2)\,dz$$

integral on RHS is exact differential

$$= \left[\int_{y,z\ \text{const}} (y^2\cos x + z^3)\,dx + \int -4\,dy + \int 2\,dz \right]_{(0\ 1\ -1)}^{(\frac{\pi}{2}\ -1\ 2)}$$

$z = \text{const}$

$$= \left[y^2 \int \cos x\,dx + z^3 \int dx + (-4)\int dy + 2\int dz \right]_{(0\ 1\ -2}^{(\frac{\pi}{2}\ -1\ 2)}$$

$$= \left[y^2(-\sin x) + z^3 x - 4y + 2z \right]_{(0\ 1\ -2)}^{(\frac{\pi}{2}\ -1\ 2)}$$

$$= 15 + 4\pi$$

Ex 14 Find the work done by the force $\bar{F} = 3x^2 i + (2xz - y)j + zk$ along the curve $x^2 = 4y$, $3x^3 = 8z$ from $x = 0$ to $x = 2$.

We have work done $W = \int_c \bar{F}\cdot d\bar{r}$

$$= \int_c 3x^2\,dx + (2xz - y)\,dy + z\,dz$$

Given curve is $x^2 = 4y$, $3x^3 = 8z$

The parametric eqn of curve is given by

$x = 2t,\ y = t^2,\ z = 3t^3$

$dx = 2dt \quad dy = 2dt \quad dz = 9t^2 dt$ and t varies from

0 to 1.

$$W = \int_0^1 3(2t)^2\, 2dt + [2(2t)(3t^3) - t^2](2t)dt + (3t^3)(9t^2)dt$$

$$= \int_0^1 (51t^5 - 2t^3 + 24t^2)\,dt = \left[51\frac{t^6}{6} - 2\frac{t^4}{4} + 24\frac{t^3}{3}\right]_0^1$$

$$= 16.$$

Ex 15. Find the work done by the force $(x^2-yz)i + (y^2-xz)j + (z^2-xy)k$ in taking particle from $(1\,1\,1)$ to $(2\,2\,0)$.

Here the path joining the points is not mentioned.

Workdone $W = \int_c \bar{F}\cdot d\bar{r}$

Now Here $\nabla \times \bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2-yz & y^2-xz & z^2-xy \end{vmatrix} = 0$

$$\therefore W = \int_c \bar{F}\cdot d\bar{r} = \int_c (x^2-yz)dx + (y^2-xz)dy + (z^2-xy)dz$$

$$= \left[\int_{yz\text{-const}} (x^2-yz)\,dx + \int_{z=\text{const}} y^2\,dy + \int z^2\,dz\right]_{(1\,1\,1)}^{(2\,2\,0)}$$

$$= \left[\frac{x^3}{3} - xyz + \frac{y^3}{3} + \frac{z^3}{3}\right]_{(1\,1\,1)}^{(2\,2\,0)} = \frac{16}{3}$$

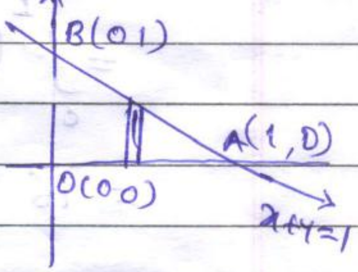
$$= \left[\cancel{\frac{x^3}{3} - xyz - xy - xyz}\right]$$

$$= \left[\frac{x^3}{3} + \frac{y^3}{3} - \frac{z^3}{3} - xyz\right]_{(1,1,1)}^{(2,2,0)} = \frac{16}{3}$$

Verify Greens lemma in the first plane for $\oint_c (3x^2-8y^2)dx + (4y-6xy)dy$ Where c is the boundary defined by $x=0,\ y=0,\ x+y=1$

By Greens lemma

$$\oint_c u\,dx + v\,dy = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$$



$u = 3x^2-8y^2,\quad v = 4y-6xy$

$$\oint_c u\,dx + v\,dy = \oint_c (3x^2-8y^2)\,dx + (4y-6xy)\,dy$$

$$= \int_{OA} (3x^2-8y^2)dx + (4y-6xy)\,dy + \int_{AB} (3x^2-8y^2)dx + (4y-6xy)dy$$

$$+ \int_{BO} (3x^2-8y^2)dx + (4y-6xy)dy$$

$$= I_1 + I_2 + I_3$$

~~Alon~~ Along OA, $y=0$, $dy=0$ & x varies from 0 to 1

$$I_1 = \int_0^1 (3x^2-0)\,dx + 0 = 3\left[\frac{x^3}{3}\right]_0^1 = 1$$

$$I_2 = \int_{AB} (3x^2-8y^2)dy + (4y-6xy)dy$$

eq^n of line is $x+y=1$ $\quad y = 1-x,\quad dy = -dx$

Ex Find the work done by the force $\overline{F} = 3x^2 i + (2xz-y) j + z k$ along the curve $x^2 = 4y$, $3x^3 = 8z$ from $x=0$ to $x=2$

We have, Work done $W = \int_c \overline{F} \cdot d\overline{r}$

$$= \int_c 3x^2 dx + (2xz - y)dy + z dz$$

Given curve is $x^2 = 4y$, $3x^3 = 8z$

The parametric eqn of curve are given by

$x = 2t$, $y = t^2$, $z = 3t^3$

$dx = 2dt$, $dy = 2t\,dt$ $dz = 9t^2$ and t varies from 0 to 1

$$W = \int_0^1 3(2t)^2 2dt + [2(2t)(3t^3) - t^2](2t)dt + (3t^3)(9t^2)dt$$

$$= \int_0^1 (51t^5 - 2t^3 + 24t^2) dt = \left[51\frac{t^6}{6} - 2\frac{t^4}{4} + 24\frac{t^3}{3}\right]_0^1$$

$= 16$

Ex Find the work done by the force $(x^2 - yz) i + (y^2 - xz) j + (z^2 - xy) k$ in taking particle from $(1, 1, 1)$ to $(2, 2, 0)$

Here path joining given points is not mentioned

~~Workdone $W = \int_c \overline{F} \cdot d\overline{r}$~~

Now Here $\nabla \times \overline{F} = \left| \quad \right| = 0$

$$\therefore W = \int_c \overline{F} \cdot d\overline{r} = \int_c (x^2 - yz)dx + (y^2 - xz)dy + (z^2 - xy)dz$$

$$= \int_{yz = \text{const}} (x^2 - yz)\,dx + \int_{z = \text{const}} -z\,dy + \int z^2\,dz$$

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x varies from 1 to 0.

$$I_2 = \int_1^0 [3x^2 - 8(1-x)^2]\,dx + [4(1-x) - 6x(1-x)(-dx)]$$

$$= \int_1^0 [3x^2 - 8(1-x)^2]\,dx + (4-10x+6x^2)(-dx)$$

$$= \int_1^0 [-3x^2 + 10x - 4 - 8(1-x^2)]\,dx$$

$$= \left[-3\left(\frac{x^3}{3}\right) + 10\left(\frac{x^2}{2}\right) - 4x - 8\frac{(1-x)^3}{-3}\right]_1^0$$

$$= \frac{8}{3}$$

$$I_3 = \int_{BO} (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$$

along BO, $x = 0$, $dx = 0$ and y varies from 1 to 0

$$I_3 = \int_1^0 0 + 4y\,dy = 4\left[\frac{y^2}{2}\right]_1^0 = -2$$

$$\int_c u\,dx + v\,dy = 1 + \frac{8}{2} - 2 = \frac{5}{3} \quad (1)$$

How $\iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx = \iint_R [-6y - (0 - 16y)]\,dx\,dy$

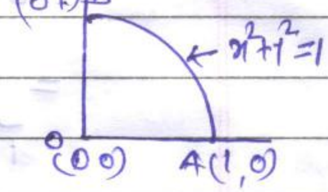
$$= \int_0^1 \int_0^{1-x} 10y\,dy\,dx = \int_0^1 10\left[\frac{y^2}{2}\right]_0^{1-x}$$

$$= \int_0^1 5(1-x)^2\,dx = 5\left[\frac{(1-x)^3}{3}\right]_0^1 = \frac{5}{3} \quad (2)$$

Hence from (1) & (2) Greens Theorem is verified

Verify Green's theorem for $\overline{F} = x\,i + y^2 j$ over the first quadrant of the circle $x^2+y^2=1$.

$$\oint_C u\,dx + v\,dy = \oint_C x\,dx + y^2\,dy$$



C is closed curve which is part of circle in first quadrant & x & y axis

$$\oint_C x\,dx + y^2\,dy = \int_{OA} x\,dx + y^2\,dy + \int_{AB} x\,dx + y^2\,dy + \int_{BO} x\,dx + y^2\,dy$$

along line OA, $y=0$, $dy=0$ & x varies from 0 to 1

along arc AB, $x = \cos\theta$, $y = \sin\theta$

$dx = -\sin\theta \cdot d\theta \quad dy = \cos\theta\, d\theta, \quad \theta : 0 \to \pi/2$

along line BO, $x=0$, $dx=0$ & y varies from 1 to 0

$$\oint_C x\,dx + y^2\,dy = \int_0^1 x\,dx + \int_0^{\pi/2} (\cos\theta)(-\sin\theta\, d\theta) + (\sin\theta)^2(\cos\theta\, d\theta) + \int_1^0 y^2\,dy$$

$$= \left[\frac{x^2}{2}\right]_0^1 + \int_0^{\pi/2} -\sin\theta \cdot \cos\theta\, d\theta + \int_0^{\pi/2} \sin^2\theta \cos\theta\, d\theta + \left[\frac{y^3}{3}\right]_1^0$$

$$= \left(\frac{1}{2} - 0\right) + \left[-\left(\frac{1 \times 1}{2}\right) + \left(\frac{1 \times 1}{3 \times 1}\right)\right] + \left(0 - \frac{1}{3}\right)$$

$$= \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{3} = 0$$

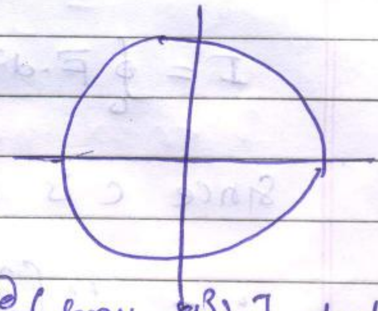
$$\& \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy = \iint_R \left(\frac{\partial}{\partial x}(y^2) - \frac{\partial}{\partial y}(x)\right) dx\,dy$$

$$= 0.$$

Hence Greens Theorem is verified.

Evaluate $\int_c (\sin y - y^3)\,dx + (xy^2 + x\cos y)\,dy$ by using Green's Theorem where c is the circle $x^2+y^2=a^2$

By Greens theorem



$$\oint_c (\sin y - y^3)\,dx + (xy^2 + x\cos y)\,dy$$

$$= \iint_R \left[\frac{\partial}{\partial x}(xy^2 + x\cos y) - \frac{\partial}{\partial y}(\sin y - y^3)\right] dx\,dy$$

$$= \iint_R \left[(y^2 + \cos y) - \cos y + 3y^2\right] dx\,dy$$

$$= \iint_R 4y^2\,dx\,dy$$

Transforming the variables to polar co-ordinates

$x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$

$\theta : 0 \to 2\pi$, $r : 0$ to a

$$\therefore \int (\sin y - y^3)\,dx + (xy^2 + x\cos y)\,dy$$

$$= \int_0^a \int_0^{2\pi} 4 \cdot r^2 \sin^2\theta \cdot r\,dr\,d\theta$$

$$= 4 \int_0^a \int_0^{2\pi} r^3 \sin^2\theta \cdot d\theta\,dr = 4\int_0^a r^3\,dr \int_0^{2\pi} \sin^2\theta\,d\theta$$

$$= 4\cdot\left[\frac{r^4}{4}\right]_0^a \left[\frac{1}{2}\right]\pi = \pi a^4$$

Ex A Vector field $F = \cos y\, i + x(1-\sin y)\, j$. Evaluate $\oint_c \bar{F}\cdot d\bar{r}$ where c is the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$ $z=0$

$$I = \oint_c \bar{F}\cdot d\bar{r} = \int_c \cos y\, dx + x(1-\sin y)\, dy$$

Since c is closed curve, We apply Green theorem

$$\therefore I = \iint_R \left[\frac{\partial}{\partial x}(x(1-\sin y)) - \frac{\partial}{\partial y}\cos y\right] dx\, dy$$

$$= \iint (1-\sin y + \sin y)\, dx\, dy = \iint_R dx\, dy$$

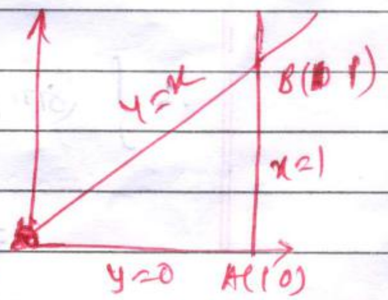
= Area bounded by ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$; $z=0$

$$= \pi(5)(2) = 10\pi.$$

Using Greens theorem evaluate $\oint_c (xy-x^2)\,dx + x^2 dy$ along the curve c formed by $x=1$, $y=0$, $y=x$

By Greens Theorem

$$\oint_c u\,dx + v\,dy = \iint\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)dx\,dy$$



$$\oint_c (xy-x^2)\,dx + x^2 dy$$

$$= \iint_R \left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(xy-x^2)\right] dx\,dy$$

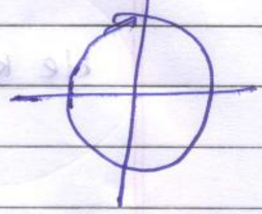
$$= \iint_R (2x-x)\,dx\,dy = \iint_R x\,dx\,dy$$

$$= \int_0^1\int_0^x x\,dx\,dy = \int_0^1 x[y]_0^x\,dx = \int_0^1 x^2 dx = \left[\frac{x^3}{3}\right]_0^1$$

$$= \frac{1}{3}.$$

Use Greens Theorem to evaluate $\oint_C \bar{F}\cdot d\bar{r}$. Where $\bar{F} = y^3\hat{i} - x^3\hat{j}$ and c is the circle $x^2+y^2=a^2,\ z=0$

<u>Sol</u> consider $I = \int_C \bar{F}\cdot d\bar{r} = \int_C y^3dx - x^3dy$



$$= \iint_R \left[\frac{\partial}{\partial x}(x^3) - \frac{\partial}{\partial y}(y^3)\right] dx\,dy$$

$$= \iint_R (-3x^2-3y^2)\,dx\,dy = -\iint_R (3x^2+3y^2)\,dx\,dy$$

R is region bounded by the circle $x^2+y^2=a^2,\ z=0$

Transforming to polar co-ordinates

$x = r\cos\theta \qquad y = r\sin\theta \qquad dx\,dy = r\,dr\,d\theta$

$r : 0$ to a & $\theta : 0$ to 2π

$$I = -\int_0^1\int_0^{2\pi} (3r^2\cos^2\theta + 3r^2\sin^2\theta)\, r\,dr\,d\theta$$

$$= -\int_0^1\int_0^{2\pi} 3r^3\,dr\,d\theta = -\int_0^1 3r^3\,dr \cdot \int_0^{2\pi} d\theta$$

$$= -3\left[\frac{r^4}{4}\right]_0^1\left[\theta\right]_0^{2\pi} = -\frac{3}{4}\cdot[2\pi] = -\frac{3\pi}{2}$$

<u>Ex</u> Using Greens Theorem, show that the area bounded by simple closed curve C is given by $\frac{1}{2}\int x\,dy - y\,dx$. Hence find the area of the circle $x^2+y^2=a^2,\ z=0$.

$$I = \frac{1}{2}\oint_C x\,dy - y\,dx = \frac{1}{2}\oint_C -y\,dx + x\,dy, \quad u=-y,\ v=x$$

Since C is simple closed curve, by Greens theorem

$$I = \frac{1}{2}\iint_R \left[\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y)\right] dx\,dy = \frac{1}{2}\iint_R 2\,dx\,dy$$

$= \iint_R dx\,dy$ = Area bounded by given closed curve

We know that $\text{Area} = \iint_R dx\,dy$

$\text{Area} = \frac{1}{2}\oint_c x\,dy - y\,dx$ By above Result

Here circle c is $x^2+y^2=a^2,\ z=0$

$x = a\cos\theta,\quad y = a\sin\theta,\quad z=0$

$dx = -a\sin\theta\,d\theta \quad dy = a\cos\theta\,d\theta \quad dz=0$

$$I = \frac{1}{2}\int_0^{2\pi} (a\cos\theta)(a\cos\theta)\,d\theta - (a\sin\theta)(-a\sin\theta)\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} a^2\,d\theta = \frac{a^2}{2}[\theta]_0^{2\pi} = \pi a^2.$$

And the work done in moving a particle once around the circle $x^2+y^2=a^2,\ z=0$ under the field of force.

$\bar{F} = \sin y\ i + x(1+\cos y)\,j$

$$W = \oint_C F\cdot d\bar{r} = \oint_C \sin y\,dx + x(1+\cos y)\,dy$$

$$= \iint_R \left[\frac{\partial}{\partial x}[x(1+\cos y)] - \frac{\partial}{\partial y}(\sin y)\right] dx\,dy$$

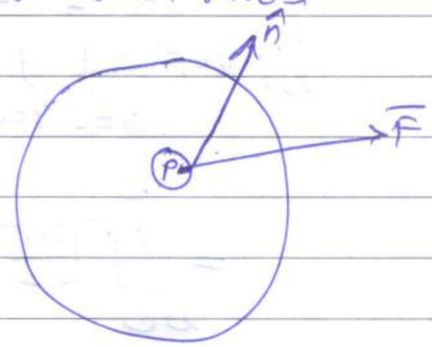
$$= \iint_R (1+\cos y - \cos y)\,dy = \iint_R dx\,dy$$

= Area bounded by circle $x^2+y^2=a^2$

$= \pi a^2$

Surface Integral : The surface integral of vector point function $\bar{F}$ over a surface S is defined as the integral of the normal component of $\bar{F}$ taken over the surfaces S

Consider a surface S. let $\bar{F}$ act at P enclosed by an element of area ds. $\hat{n}$ is a unit vector normal to the surface at P. Normal component of $\bar{F}$ is given by $\bar{F}\cdot\hat{n}$



The surface integral can be expressed as

$$\int_S \bar{F}\cdot\hat{n}\,ds \quad \text{or} \quad \iint_S (\bar{F}\cdot\hat{n})\,ds$$

If we write $d\bar{s} = \hat{n}\,ds$, the above integral can also be written as $\int_S \bar{F}\cdot d\bar{s}$ or $\iint_S \bar{F}\cdot d\bar{s}$

Gauss Divergence Theorem

Gauss divergence theorem states that the surface integral of the normal component of a vector point function $\bar{F}$ over a closed surface S is equal to the volume integral of a divergence of $\bar{F}$ taken throughout the volume V enclosed by the surface S

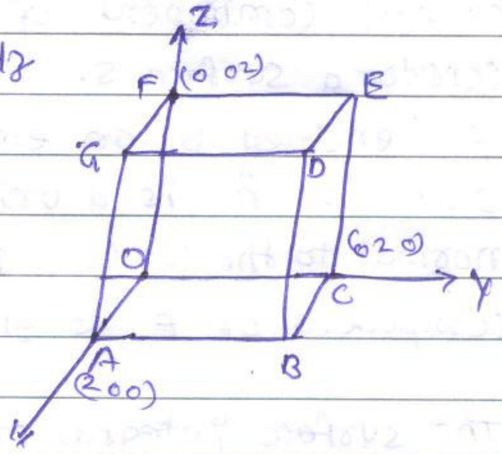
It is written as

$$\iint_S \bar{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\bar{F}\,dV.$$

Ex 1 Verify divergence theorem for $\bar{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ and S, the surface of the cube bounded by the planes $x=0,\ x=2,\ y=0,\ z=0,\ z=2\ y=2$.

Taking co-ordinate axes as shown in fig. we proceed to evaluate volume & surface integrals.

$$\nabla\cdot\bar F = \frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz) = 4z - 2y + y$$
$$= 4z - y$$



$$\iiint \nabla\cdot\bar F = \int_{x=0}^{2}\int_{y=0}^{2}\int_{z=0}^{2} (4z-y)\,dx\,dy\,dz$$

$$= \iint \left[\frac{4z^2}{2} - yz\right]_0^2 dx\,dy$$

$$= \int_0^2\int_0^2 (8-2y)\,dx\,dy$$

$$= \int_0^2 \left[8y - \frac{2y^2}{2}\right]_0^2 dx = \int_0^2 (16-4)\,dx = 12[x]_0^2 = 24$$

$$\boxed{\iiint_V \nabla\cdot\bar F = 24}$$

Now to evaluate the surface integral, consider surfaces $S_1 = OABC$, $S_2 = GDEF$, $S_3 = OAGF$, $S_4 = BCED$
$S_5 = OCEF$ $S_6 = ABDG$

$$I_1 = \iint_{S_1} \bar F\cdot\hat n\,ds$$

For S_1 $\hat n = -k$, $ds = dx\,dy$

$$\bar F\cdot\hat n = \bar F\cdot(-k) = (4xz\,i - y^2 j + yz\,k)\cdot(-k)$$
$$= -yz$$

$I_1 = \iint -yz\,dx\,dy$ But $z=0$ in the plane OABC
$= 0$

For S_2 $\hat n = k$ $ds = dx\,dy$ $\bar F\cdot\hat n = yz$

$$I_2 = \iint yz\,dx\,dy = 2\int_0^2\int_0^2 y\,dx\,dy = 2\int_0^2 \left[\frac{y^2}{2}\right]_0^2 dx$$

$= 4[x]_0^2 = 8$

For S_3 ie surface OAGF , $\hat{n} = -\hat{j}$

$ds = dx\,dy \qquad F \cdot \hat{n} = F \cdot (-\hat{j}) = y^2$

$I_3 = \iint y^2 dx\,dy$ But $y = 0$ in the plane S_3

$I_3 = 0$

For S_4 ie surface BCED, $\hat{n} = \hat{j}, ds = dx\,dy$

$F \cdot \hat{n} = -y^2 \quad , \quad I_4 = \int_0^2\int_0^2 -y^2 dx\,dz$

but $y = 2$,

$= -4\int_0^2 [z]_0^2 dx = -4\int_0^2 2\,dx = -16$

for S_5 ie surface OCEF, $\hat{n} = -\hat{i}$

$ds = dy\,dz \qquad F \cdot \hat{n} = F \cdot (-\hat{i}) = -4xz$

$I_5 = \int_0^2\int_0^2 -4xz\,dy\,dz$ But $x = 0$ in this plane

$= 0$

lastly for the surface S_6. ABDG, $\hat{n} = \hat{i}$

$F \cdot \hat{n} = F \cdot \hat{i} = 4xz$ & $ds = dy\,dz$, $x = 2$

$I_6 = \int_0^2\int_0^2 4xz\,dy\,dz = \int_0^2\int_0^2 8z\,dy\,dz$

$= 8\int_0^2 \left[\frac{z^2}{2}\right]_0^2 dy = 16[y]_0^2 = 32$

The surface integral which is sum of all these integrals

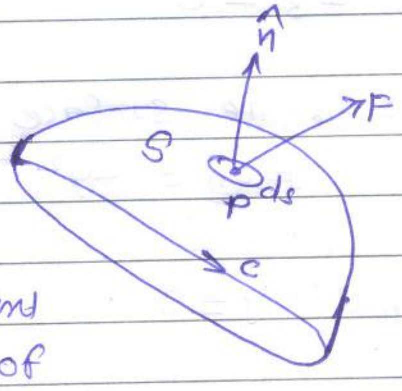
$= I_1 + I_2 + I_3 + I_4 + I_5 + I_6$

$= 0 + 8 + 0 - 16 + 0 + 32 = 24$

$\therefore \iint_S F \cdot \hat{n}\, ds = 24 = \iiint_V \nabla \cdot F\, dV$

Stokes Theorem & Related problems

The surface integral of the normal component of the curl of the vector point function $\bar{F}$ taken over an open surface S bounded by closed curve C is equal to the line integral of the tangential component of $\bar{F}$ taken around the curve C.



In fig. S is the open surface to which $\hat{n}$ is unit outward drawn normal vector. $\bar{F}$ is acting at P enclosed by element ds. Curve C is the boundary of surface S.

The stokes theorem can be expressed as

$$\iint_S \hat{n} \cdot \text{curl}\, \bar{F}\, ds = \oint_C \bar{F} \cdot d\bar{r}$$

(OR)

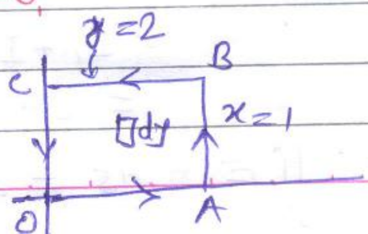
$$\int_S \hat{n} \cdot \text{curl}\, \bar{F}\, ds = \oint_C \bar{F} \cdot d\bar{r}$$

$$\oint_C \bar{F} \cdot d\bar{r} = \iint_S \nabla \times \bar{F} \cdot d\bar{S}$$

Ex Verify stokes theorem for

$$\bar{F} = xy^2\,\hat{i} + y\,\hat{j} + z^2 x\,\hat{k}$$

for the surface of rectangular lamina bounded by $x=0,\ y=0,\ x=1,\ y=2,\ z=0$.



$\Rightarrow\ \bar{F} = xy^2\,\hat{i} + y\,\hat{j} + z^2 x\,\hat{k},\ z=0$

$\int_C \bar{F} \cdot d\bar{r} = \int_C xy^2\,dx + y\,dy$

Where c is the path OABCO as shown in fig.

along OA, $y=0$ $dy=0$

along AB $x=1$ $dx=0$

along BC $y=2$ $dy=0$

along CO $x=0$ $dx=0$

$$\int_c \bar{F}\cdot d\bar{r} = \int_{OA} xy^2 dx + \int_{AB} y\,dy + \int_{BC} xy^2 dx + \int_{CO} y\,dy$$

$$= 0 + \int_0^2 y\,dy + \int_1^0 4x\,dx + \int_2^0 y\,dy$$

$$= \int_0^2 y\,dy + 4\left[\frac{x^2}{2}\right]_1^0 - \int_0^2 y\,dy = 2\left[x^2\right]_1^0 = -2$$

To obtain surface integral

$$\nabla\times\bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy^2 & y & 0 \end{vmatrix} = i(0)+j(0)+k(-2xy)$$

normal to the surface $\hat{n} = k$, $ds = dx\,dy$ is surface element in S

$$\iint_S (\nabla\times\bar{F})\cdot\hat{n}\,ds = \iint_S (-2xy)k\cdot k\,dx\,dy$$

$$= -2\int_{x=0}^{1}\int_0^2 xy\,dx\,dy = -2\int_0^1 x\left[\frac{y^2}{2}\right]_0^2$$

$$= -2\int_0^1 x\left(\frac{4}{2}\right)dx = -4\left[\frac{x^2}{2}\right]_0^1 = -2$$

Thus $\boxed{\int_c \bar{F}\cdot d\bar{r} = \iint_S (\nabla\times\bar{F})\cdot\hat{n}\,ds = 2}$

Stokes Theorem is verified.

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Ex 2. Apply stokes theorem to calculate $\int_C 4y\,dx + 2z\,dy + 6y\,dz$ where c is the curve of intersection of $x^2+y^2+z^2 = 6z$ and $z = x+3$.

Taking $\overline{F} = 4y\hat{i} + 2z\hat{j} + 6y\hat{k}$ & applying stokes theorem

$$\int_C 4y\,dx + 2z\,dy + 6y\,dz = \iint_S (\nabla \times \overline{F}) \cdot \hat{n}\, ds$$

where s is the surface of the surface $x^2+y^2+z^2=6z$ and $z = x+3$, $\hat{n}$ is normal to the plane $x-z+3=0$

let $\phi = x-z+3 \quad \frac{\partial \phi}{\partial x} = 1, \ \frac{\partial \phi}{\partial y} = 0, \ \frac{\partial \phi}{\partial z} = 1$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{i-k}{\sqrt{2}}.$$

$$\nabla \times \overline{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 4y & 2z & 6y \end{vmatrix} = 4\hat{i} - 4\hat{k}$$

$$(\nabla \times \overline{F}) \cdot \hat{n} = (4\hat{i} - 4\hat{k}) \cdot \left(\frac{i-k}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}(4+4) = 4\sqrt{2}$$

$$\iint_S (\nabla \times \overline{F}) \cdot \hat{n}\, ds = \iint_S 4\sqrt{2}\, ds = 4\sqrt{2} \times \text{area of circle.}$$

Centre of the sphere $x^2+y^2+(z-3)^2 = 9$; $(0,0,3)$ lies on the plane $z = x+3$ that means given circle is a great circle of the sphere where radius is same as that of the sphere

$\therefore$ Radius of circle $= 3$, Area $= \pi(3)^2 = 9\pi$

$$\therefore \iint_S (\nabla \times \overline{F}) \cdot \hat{n}\, ds = 4\sqrt{2} \times 9\pi = 36\pi\sqrt{2}.$$