Course: Integral Calculus and Differential Equations (23UBSL1201)

Class: FY B Tech (Term-2, 2023-24)

Question Banks

Unit -1 Integral Calculus

- **Curve Tracing**

1. Trace the curve

   a. $y^2(2a-x)=x^3$

   b. $x^2y^2=a^2(y^2-x^2)$

   c. $y^2(a+x)=x^2(a-x)$

   d. $a\,y^2=x^2(a-x)$

- **Gamma Function:**

Type-1:

Evaluate $\int_0^\infty \sqrt[4]{x}\, e^{-\sqrt{x}}\, dx$

Show that $\int_0^\infty \sqrt{x}\, e^{-x^3}\, dx = \frac{\sqrt{\pi}}{3}$

Evaluate $\int_0^\infty x^7 e^{-2x^2}\, dx$

Evaluate $\int_0^\infty \sqrt{y}\, e^{-\sqrt{y}}\, dy$

Prove that $\int_0^\infty \sqrt{x}\, e^{-\sqrt[3]{x}}\, dx = \frac{316\sqrt{\pi}}{16}$

Evaluate $\int_0^\infty x^9 e^{-2x^2}\, dx$

Evaluate $\int_0^\infty x^n e^{-2x^m}\, dx$

Type-2

Evaluate $\int_0^\infty \frac{x^a}{a^x}\, dx \quad (a>0)$

Evaluate $\int_0^\infty \frac{dx}{3^{4x^2}}$

Evaluate $\int_0^\infty \frac{x^3}{3^x}\, dx$

Evaluate $\int_0^\infty \frac{x^3}{3^{x^2}}\, dx$

Prove that $\int_0^\infty \frac{x^4}{4^x}\, dx = \frac{24}{(\log 4)^5}$

Evaluate $\int_0^\infty \frac{x^n}{a^{x^m}}\, dx$

Type-3 (Examples involving log ax

Evaluate $\int_0^1 x^m (\log x)^n\, dx$

Prove that $\int_0^1 \frac{dx}{\sqrt{x\log\frac{1}{x}}} = \sqrt{2\pi}$

Evaluate $\int_0^1 (x\log x)^4\, dx$

Prove that $\int_0^1 \frac{dx}{\sqrt{-\log x}} = \sqrt{\pi}$

Evaluate $\int_0^1 x^{a-1}\left(\log\frac{1}{x}\right)^{n-1} dx;\ a>0$

- **Beta Functions:**

Type-1 Examples by using Definition:

Prove that $\int_0^1 x^3(1-\sqrt{x})^5\, dx = \frac{1}{5148}$

Evaluate $\int_0^{2a} x^{7/2}(2a-x)^{-1/2}\, dx$

Show that $\int_0^2 x(8-x^3)^{1/3}\, dx = \frac{16\pi}{9\sqrt{3}}$

Evaluate $\int_0^{2a} x\sqrt{2ax-x^2}\, dx$

Show that $\int_0^1 \frac{dx}{\sqrt[3]{1-x^3}} = \frac{2\pi}{3\sqrt{3}}$

Evaluate $\int_0^1 x^m(1-x^n)^p\, dx$

Evaluate $\int_0^n x^n(n-x)^p\, dx$

Evaluate $\int_0^4 \sqrt{x}\,(4-x)^{3/2}\, dx$

Type-2 Examples using Formula $\int_0^{\pi/2} \sin^p\theta \cos^q\theta\, d\theta = \frac{1}{2} B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$

Prove that $\int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} \cdot \int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta = \pi$

Prove that $\int_0^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}$

Show that $\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta \cdot \int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta = \frac{\pi^2}{2}$

Type-3 Examples having general form $\int_a^b (x-a)^m (b-x)^n\, dx$

Evaluate $\int_2^5 (x-2)^3 (5-x)^2\, dx$

Evaluate $\int_3^5 (x-3)^{1/2} (5-x)^{1/2}\, dx$

Evaluate $\int_3^7 (x-3)^{1/4} (7-x)^{1/4}\, dx$

Type-4 Examples using $\int_0^{\infty} \frac{x^m}{(1+x)^{m+n}}\, dx$

Evaluate $\int_0^{\infty} \frac{x^8(1-x^6)}{(1+x)^{24}}\, dx$

Prove that $\int_0^1 \frac{y^{m-1} + y^{n-1}}{(1+y)^{m+n}}\, dx = B(m,n)$

Evaluate $\int_0^{\infty} \frac{x^9(1-x^5)}{(1+x)^{25}}\, dx$

- **DUIS Rule-1**

Verify DUIS Rule – 1 for $I(a) = \int_0^{\pi/2} \sin ax\, dx$

Prove that $\int_0^1 \frac{x^a - x^b}{\log x}\, dx = \log\left(\frac{a+1}{b+1}\right)$

Evaluate $\int_0^{\infty} \frac{e^{-x}}{x}(1-e^{-ax})dx,\ a > -1$

Prove that $\int_0^1 \frac{x^a - 1}{\log x}\, dx = \log(a+1),\ a \geq 0$

Show that $\int_0^{\infty} \frac{e^{-ax} - e^{-bx}}{x}\, dx = \log\left(\frac{b}{a}\right)$

- **DUIS Rule-2**

Verify DUIS Rule-2 for

1. $I(a) = \int_a^{a^2} \frac{1}{x+a}\, dx$

2. $I(a) = \int_a^{a^2} \tan^{-1}\left(\frac{x}{a}\right) dx$

3. $I(a) = \int_a^{a^2} \log(ax)\, dx$

4 Find $\frac{dI(a)}{da}$ if $I(a) = \int_{\pi/6a}^{\pi/2a} \frac{\sin ax}{x}\, dx$

Unit -2 Multiple Integral

Example on Direct Evaluation of Double Integral

A. Evaluate the following

1) $\int_0^1 \int_{x^2}^{2-x} y \, dy \, dx$ 2) $\int_0^1 \int_0^y xy \, dx \, dy$ 3) $\int_0^1 \int_0^{1-x} (x+y) dy \, dx$

4) $\int_0^1 \int_0^x (x^2 + 3y) \, dy \, dx$ 5) $\int_0^1 \int_0^1 \frac{dx \, dy}{\sqrt{(1-x^2)(1-y^2)}}$

B. Show that

1) $\int_0^1 \int_0^1 \frac{dx \, dy}{\sqrt{(1-x^2)(1-y^2)}} = \frac{\pi^2}{4}$ 2) $\int_0^1 \int_{-\sqrt{y}}^{-y^2} xy \, dx \, dy = \frac{-1}{12}$

3) $\int_0^1 \int_{x^2}^{x} xy(x+y) \, dx \, dy = \frac{3}{56}$ 4) $\int_0^1 \int_x^{\sqrt{x}} (x^2 + y^2) \, dx \, dy = \frac{3}{15}$

Examples: Integrals when limits are not provided

1. Evaluate $\iint x^2 y^2 \, dx \, dy$ over the positive quadrant of the circle $x^2 + y^2 = 1$.
2. Evaluate $\iint \frac{1}{(x^4+y^2)} \, dx \, dy$ over region $y \geq x^2, x \geq 1$.
3. Evaluate $\iint_R \sqrt{xy(1-x-y)} \, dx \, dy$, where R is the area bounded by $x = 0$, $y = 0$ and $x + y = 1$.
4. Evaluate $\iint y \, dx \, dy$, over the area bounded by $y = x^2$ and $x + y = 2$.
5. Evaluate $\iint y \, dx \, dy$, over the area bounded by $y = x^2$ and $y = x$.
6. Evaluate $\iint_R y \, dxdy$, where R is $y = x^2$ and $x + y = 2$ in first quadrant.

Change of Order of Integration

1. Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \frac{dx \, dy}{(1+e^y)\sqrt{1-x^2-y^2}}$
2. Show that $\int_0^a \int_{\frac{y^2}{a}}^{y} \frac{y \, dx \, dy}{(a-x)\sqrt{ax-y^2}} = \frac{\pi a}{2}$
3. Evaluate $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} dx \, dy$ by changing the order of integration
4. Evaluate $\int_0^1 \int_{x^2}^{2-x} xy \, dx \, dy$ by changing the order of integration
5. Evaluate $\int_0^\infty \int_{x^2}^{x} x \, e^{-x^2/y} \, dy \, dx$ by changing the order of integration

Unit -3: Vector Integral Calculus

A. Line Integral / Work Done

1. Evaluate $\int_c \bar{F} \cdot d\bar{r}$ for $\bar{F} = 3x^2 i + (2xz - y)j + z\,k$ along the following paths
   a. the straight line joining the points (0,0,0) and (2,1,3).
   b. Along the curve $x = 2t^2, y = t, z = 4t^2 - t$ from $t = 0$ to $t = 1$.
   c. Along the curve defined by $x^2 = 4y, 3x^3 = 8z$ from $x = 0$ to $x = 2$
2. Evaluate $\int_c \bar{F} \cdot d\bar{r}$ for $\bar{F} = (2x + y)i + (3y - x))j$ and c is the curve:
   a. the straight line joining the points (0,0) and (3, 2).
   b. Along the path joining (0,0)and (2,0) and then from (2,0) to (0,3).
3. Evaluate $\int_c \bar{F} \cdot d\bar{r}$ for $\bar{F} = (2x + 3)i + xyj + (yz - x)j$ along the following paths:
   a. the straight lines from (0,0,0) to (0,0,1) then to (0.1,1) and then to (2,1,1).
   b. The straight line joining (0, 0, 0) and (3, 1, 1,).
4. Find the work done in moving a particle once round the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1, z = 0$ under the field of force given by $\bar{F} = (2x - y + z)i + (x + y - z^2)j + (3z - 2y + 4z)\,k$.
5. Find the work done in moving a particle once round the ellipse $\frac{x^2}{16} + \frac{y^2}{4} = 1, z = 0$ under the field of force given by $\bar{F} = (2x - y + z)i + (x + y - z^2)j + (3z - 2y + 4z)\,k$
6. Find the work done in moving a particle from $(0,1,-1)$ to $\left(\frac{\pi}{2}, -1, 2\right)$ in a force field $\bar{F} = (y^2 \cos x + z^3)i + (2y \sin x - 4)j + (3x z^2 + 2)k$
7. Find the work done by the force $(x^2 - yz)i + (y^2 - xz)j + (z^2 - xy)k$ in taking particle from (1, 1, 1) to (2, 2, 0).

B. Greens theorem

8. Verify Greens theorem for $\bar{F} = x\,i + y^2 j$ over the first quadrant of the circle $x^2 + y^2 = a^2$
9. Evaluate $\oint_c [\cos y\, i + x(1 - \sin y)j] \cdot d\bar{r}$ for the closed curve which is given by $x^2 + y^2 = 1,\ z = 0$.
10. Verify Greens theorem for $\bar{F} = x^2 i + xy\,j$ over the region R bounded by $y = x^2$ and the line $y = x$.
11. Verify Greens theorem for $\bar{F} = x\,i + y^2 j$ over the first quadrant of the circle $x^2 + y^2 = 1$
12. If $\bar{F} = \frac{1}{x^2+y^2}[-yi + x\,j]$ then show that $\oint_c \bar{F} \cdot \overline{dr} = 2\pi$, where c is the circle containing origin.
13. Evaluate $\oint_c (\sin y - y^3)dx + (xy^2 + x \cos y)dy$ by using Green's theorem , Where C is the circle $x^2 + y^2 = a^2$

C. Stokes Theorem

14. Verify Stokes theorem for $\bar{F} = xy^2 i + y\,j + x\,z^2 k$ for the surface of rectangular lamina bounded by $x = 0,\ y = 0, x = 1, y = 2, z = 0$.
15. Apply Stokes theorem to calculate $\int_c (4y\,dx + 2z\,dy + 6y\,dz)$ where c is the curve of the intersection of $x^2 + y^2 + z^2 = 6z$ and $z = x + 3$.
16. Verify Stokes theorem for $\bar{F} = (2x - y)i - y\,z^2 j - z\,y^2 k$ and S is the surface of Hemisphere $x^2 + y^2 + z^2 = 1$ above XOY plane.
17. *Evaluate* $\iint_S (\nabla \times \bar{F})^\circ \hat{n}\, ds$ for $\bar{F} = (x^2 + y - 4)i + 3xy\,j + (2xz + z^2)k$ over the surface S of Hemisphere $x^2 + y^2 + z^2 = 16$ above XOY plane.

Unit-4: Differential Equations

Linear Differential Equations

1. Solve $(1+y^2)+\left(x-e^{-\tan^{-1}y}\right)\frac{dy}{dx}=0$
2. Solve $x^2(x^2-1)\frac{dy}{dx}+x(x^2+1)y=x^2-1$
3. Solve $(1+\sin y)\frac{dx}{dy}=2y\cos y-x(\sec y+\tan y)$
4. Solve $y^2+\left(x-\frac{1}{y}\right)\frac{dy}{dx}=0$
5. Solve $(x^2+1)\frac{dy}{dx}+4xy=\frac{1}{(x^2+1)^2}$
6. Solve $(e^{-y}\sec^2 y-x)dy=dx$
7. $(1+x^2)\frac{dy}{dx}+xy=1$

Equations Reducible to Linear

1. Solve: $\sin y\ \frac{dy}{dx}=\cos x\ (2\cos y-\sin^2 x)$
2. Solve: $\frac{dy}{dx}-\frac{\tan y}{1+x}=(1+x)e^x\sec y$
3. Solve: $\cos y\ -x\sin y\ \frac{dy}{dx}=\sec^2 x$
4. Solve: $\frac{dy}{dx}=\frac{y^3}{e^{2x}+y^2}$
5. Solve: $xy-\frac{dy}{dx}=y^3\ e^{-x^2}$
6. Solve: $\left(x\,y^2+e^{-\frac{1}{x^3}}\right)dx-x^2y\,dy=0$

Exact Differential Equations

1. Solve $(x+y-2)dx+(x-y+4)dy=0$
2. Solve $\left(\frac{y^2}{(y-x)^2}-\frac{1}{x}\right)dx+\left(\frac{1}{y}-\frac{x^2}{(x-y)^2}\right)dy=0$
3. Solve $\frac{dy}{dx}=\frac{\tan y-2xy-y}{x^2-x\tan^2 y+\sec^2 y}$
4. Solve $(y^2e^{xy^2}+4x^3)dx+(2xy\,e^{xy^2}-3y^2)dy=0$
5. Solve $\frac{dy}{dx}+\frac{y\cos x+\sin y+y}{\sin x+x\cos y+x}$
6. Solve $\frac{dy}{dx}=\frac{4x-2y+1}{2x-6y+2}$

Higher order linear differential equations with constant coefficients (Cases of CF & PI).

Solve the following Differential Equations:

1. $(D^2+2D+5)y=0$
2. $\frac{d^4y}{dx^4}-5\frac{d^2y}{dx^2}+12\frac{dy}{dx}+28y=0$
3. $(D^2+4)y=0$
4. $\frac{d^6y}{dx^6}+6\frac{d^4y}{dx^4}+9\frac{d^2y}{dx^2}=0$
5. $(D^4+2D^2+1)y=0$
6. $\frac{d^2y}{dx^2}-5\frac{dy}{dx}-6y=0$
7. $2\frac{d^2y}{dx^2}-\frac{dy}{dx}-10y=0$
8. $(D^3+D^2-2D+12)y=0$
9. $(D^3+6D^2+11D+6)y=0$
10. $\frac{d^4y}{dx^4}+m^4y=0$

Examples- General Method

Solve the following Differential equations:

1. $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{e^x}$
2. $\frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{1}{1+e^x}$
3. $(D^2 + 5D + 6)y = e^{-2x} \sec^2 x \ (1 + 2 \tan x)$
4. $\frac{d^2y}{dx^2} + 9y = \sec 3x$
5. $(D^2 - 1)y = e^{-x} \sin e^{-x} + \cos e^{-x}$

Short cut Methods:

(P. I when $f(x) = e^{ax}$)

1. Solve $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = e^{-3x}$
2. *Solve* $(D^2 - 5D + 6)y = 3e^{5x}$
3. Find the Particular integral of $(D-1)^2 y = e^x + 2^x - \frac{3}{2}$
4. Find the Particular integral of $(D-2)^2(D+1)y = e^{2x} + 2^{-x}$
5. Solve $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 3y = e^{-6x}$

(P. I when $f(x) = \sin(ax+b)$ or $\cos(ax+b)$

6. Solve $\frac{d^3y}{dx^3} + 4\frac{dy}{dx} = \sin 2x$
7. Solve $(D^2 + 2D + 1)y = 4 \sin 2x$
8. Solve $(D^2 + 1)y = \sin x \sin 2x$
9. Solve $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = \cos 3x$

(P. I when $f(x) = \sinh(ax+b)$ or $\cosh(ax+b)$

6. Solve $\frac{d^3y}{dx^3} - 4\frac{dy}{dx} = 2\cosh 2x$

10. Solve $(D^2 + 2D + 1)y = 4 \sinh 2x$

(P. I when $f(x) = x^m$)

11. Find the Particular Integral of $\frac{d^3y}{dx^3} - \frac{dy}{dx} + y = x^3 - 3x^2 + 1$
12. Solve $(D^3 - 2D + 4)y = 3x^2 - 5x + 2$

(P. I when $f(x) = e^{ax} V$ *where V is any function of x*)

13. Solve $(D^2 - 4D + 3)y = x^3 e^{2x}$
14. Solve $\frac{d^3y}{dx^3} - 7\frac{dy}{dx} - 6y = e^{2x}(1+x)$

(P. I when $f(x) = xV$ *where V is any function of x*)

15. Solve $\frac{d^2y}{dx^2} - 4y = x \sin x$
16. Solve $(D^2 - 2D + 1)y = x\, e^x \sin x$
17. Solve $(D^2 + 2D + 1)y = x\, e^{-x} \cos x$

Unit-5 Differential Equations

Method of Variation of Parameters

Solve following differential equations by method of variation of Parameters.

1. $(D^2 + 4)y = sec\ 2x$
2. $\frac{d^2y}{dx^2} + y = cosec\ x$
3. $\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$
4. $(D^2 - 2D + 2)y = e^x \tan x$
5. $(D^2 - 6D + 9)y = \frac{e^{3x}}{x^2}$

Cauchy's Or Euler's Homogeneous Linear Differential Equations:

Solve the following Differential Equations:

1. $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 4y = \cos(\log x) + x\ \sin(\log x)$
2. $x^3\frac{d^3y}{dx^3} + 2x^2\frac{d^2y}{dx^2} + 2y = 10\left(x + \frac{1}{x}\right)$
3. $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 5y = x^2\sin(\log x)$
4. $x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = x^5$

Legendre's Linear Equations:

Solve the following differential Equations:

1. $(2x+1)^2\frac{d^2y}{dx^2} - 2(2x+1)\frac{dy}{dx} - 12y = 6x$
2. $(1+x)^2\frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 2\sin[\log(1+x)]$
3. $(3x+2)^2\frac{d^2y}{dx^2} + 3(3x+2)\frac{dy}{dx} - 36y = 3x^2 + 4x + 1$
4. $(2x+1)^2\frac{d^2y}{dx^2} - 6(2x+1)\frac{dy}{dx} + 16y = 8(2x+1)^2$

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